Convective corrections to the linear diffusion equation

Seth Putterman
Department of Physics, University of California, Los Angeles, California 90049, USA

Paul Roberts
Department of Mathematics, University of California, Los Angeles, California 90049, USA
(Received 15 October 2007; published 28 April 2008)

The classical problem of the diffusion of heat in a homogeneous medium is reexamined, the medium being confined by fixed boundaries maintained at a fixed temperature. When the thermal diffusivity is small, the relaxation of the temperature of the medium to that of the boundary proceeds on two time scales, one associated with a lightly damped high-frequency acoustic mode and the other with an aperiodically damped diffusive mode. Considering for simplicity a spherical configuration, it is shown that the latter does not obey the classical linear heat conduction equation.

DOI: 10.1103/PhysRevE.77.046313 PACS number(s): 47.10.−g, 44.10.+i, 43.20.+g, 47.35.Rs

I. INTRODUCTION

The low frequency relaxation to equilibrium of a thermal disturbance in a continuous medium with stationary boundaries is generally described, in the linear approximation, by the thermal diffusion equation [[1], Eq. (50.4)],

$$\partial T/\partial t = \chi \nabla^2 T, \tag{1}$$

where T is temperature, $\chi = \kappa/\rho c_p$ is thermal diffusivity, ρ is density, κ is thermal conductivity, $c_p = T(\partial s/\partial T)_p$ is heat capacity at constant pressure p, and s is entropy per unit mass; T, p, s, and ρ are functions of position $\mathbf{x}$ and time t but deviate so slightly from constant values that the χ in Eq. (1) is constant and this is what is meant by referring to Eq. (1) as being true in the linear approximation. Nevertheless it is the equation generally found in textbooks to describe the transmission of heat.

The main purpose of this paper is to demonstrate a surprising fact: under the specified conditions, Eq. (1) does not describe long wavelength thermal diffusion correctly. As an example of the more general situations to which the analysis of this paper apply, consider a spherically symmetric thermal disturbance in a spherical region of radius R of homogeneous material bounded by stationary impermeable walls maintained at constant fixed temperature. In this case, we may use spherical coordinates (r, θ, ϕ) for position $\mathbf{x}$ so that $T = T(r,\theta,\phi,t)$; here r is distance from the center of symmetry. An analysis of the low frequency response indicates that T is governed by

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T + \frac{\alpha T}{\rho c_p}\frac{\partial p}{\partial t}, \tag{2}$$

in which $\alpha = -\rho^{-1}(\partial \rho/\partial T)_p$ is the thermal expansion coefficient. The last term, which is associated with radial motion within the region associated with thermal expansion, is in general nonzero. We show, in the case of small χ, that the last term in Eq. (2) can be transformed for the thermal diffusion modes into an expression involving T. The equation then governing T and replacing Eq. (1) is shown to be

$$\frac{\partial T}{\partial t} - \chi \nabla^2 T = \frac{3\chi(\gamma-1)}{R}\left(\frac{\partial T}{\partial r}\right)_{r=R}, \tag{3}$$

where $\gamma = c_p/c_v$ and $c_v = T(\partial s/\partial T)_\rho$ is the heat capacity at constant volume. This gives decay times that are different from the ones usually presented in textbooks (see, for example, [[1], p. 208]).

In the limit in which the thermal expansion coefficient vanishes, there is no difference between Eqs. (1) and (3). When the thermal expansion coefficient is nonzero, however, the time-varying temperature produces a time varying density that creates divergence or convergence in the velocity, and this couples to the sound field, so producing the deviation from Eq. (1) represented by the right-hand side of Eq. (3). The two physical rates that characterize the sound field and diffusion are c/R and χ/R^2, where c is the speed of sound: $c^2 = (\partial p/\partial \rho)_s$. Depending on which of these is the larger, the thermal mode has a high or low frequency character. Equation (3) applies in the low frequency limit, but both limits are examined in this paper.

A. Basic equations

We use the same notation as [1], in terms of which the conservation laws of mass, momentum, and energy are

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v}, \tag{4}$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p, \tag{5}$$

$$\rho T \frac{ds}{dt} = \nabla \cdot (\kappa \nabla T), \tag{6}$$

where $\mathbf{v}$ is the fluid velocity and $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the motional derivative. Our independent thermodynamic variables will be ρ and T. With this in mind, we observe first that

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_T dp + \left(\frac{\partial \rho}{\partial T}\right)_p dT = c_T^{-2} dp - \rho\alpha dT, \tag{7}$$

where c_T is the isothermal sound speed: $c_T^2 = (\partial p/\partial \rho)_T$. It follows that

$$\nabla p = c_T^2 \nabla \rho - c_T^2 \rho \alpha \nabla T. \tag{8}$$

Second, by a Maxwell relation, we also have

$$ds = \left(\frac{\partial s}{\partial T}\right)_\rho dT + \left(\frac{\partial s}{\partial \rho}\right)_T d\rho = \left(\frac{\partial s}{\partial T}\right)_\rho dT + \left(\frac{\partial s}{\partial p}\right)_T \left(\frac{\partial p}{\partial \rho}\right)_T d\rho$$
$$= \left(\frac{\partial s}{\partial T}\right)_\rho dT + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_p \left(\frac{\partial p}{\partial \rho}\right)_T d\rho = c_v \frac{dT}{T} - \alpha c_T^2 \frac{d\rho}{\rho}. \tag{9}$$

It follows that

$$\rho T \frac{ds}{dt} = \rho c_v \frac{dT}{dt} - \alpha T c_T^2 \frac{d\rho}{dt}. \tag{10}$$

With the help of Eqs. (4), (8), and (10), we may write Eqs. (5) and (6) as

$$\rho \frac{d\mathbf{v}}{dt} = - c_T^2 \nabla \rho + c_T^2 \rho \alpha \nabla T, \tag{11}$$

$$\rho c_v \frac{dT}{dt} = - \alpha T c_T^2 \rho \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T). \tag{12}$$

II. PERTURBATION EQUATIONS: EIGENVALUE PROBLEM

The basic state, the oscillations of which are to be studied, is defined by

$$p = p_0,$$

$$T = T_0,$$

$$\rho = \rho_0. \tag{13}$$

where p_0, T_0, and ρ_0 are constants. In what follows α, c_T, c_p, c_v, γ, and χ take their values in the basic state and are constants too.

Write

$$p = p_0 + p',$$

$$T = T_0 + T',$$

$$\rho = \rho_0 + \rho', \tag{14}$$

and substitute into Eqs. (4), (11), and (12), the linearized form of which become

$$\frac{\partial \rho'}{\partial t} = - \rho_0 \nabla \cdot \mathbf{v}, \tag{15}$$

$$\frac{\partial \mathbf{v}}{\partial t} = - \frac{c_T^2}{\rho_0} \nabla \rho' - c_T^2 \alpha \nabla T', \tag{16}$$

$$\frac{\partial T'}{\partial t} = \gamma \chi \nabla^2 T' - \frac{(\gamma - 1)}{\alpha} \nabla \cdot \mathbf{v}, \tag{17}$$

where we have used the thermodynamic relation $c_p - c_v = \alpha^2 T c_T^2$. Make the equations dimensionless by using the radius R of the basic configuration as unit of length and R/c_T, c_T, ρ_0, $c_T^2 \rho_0$, and α^{-1} as units of t, $\mathbf{v}$, ρ', p', and T'. Then $p' = \rho' + T'$. Define ϵ by

$$\epsilon^2 = \frac{\chi}{R c_T}. \tag{18}$$

From Eq. (16), we may write

$$\mathbf{v} = - \nabla \psi', \tag{19}$$

and obtain from Eqs. (15)–(19)

$$\frac{\partial \psi'}{\partial t} = \rho' + T', \tag{20}$$

$$\frac{\partial \rho'}{\partial t} = \nabla^2 \psi', \tag{21}$$

$$\frac{\partial T'}{\partial t} - \epsilon^2 \gamma \nabla^2 T' = (\gamma - 1) \nabla^2 \psi'. \tag{22}$$

Equations (20) and (21) imply

$$\frac{\partial T'}{\partial t} = \frac{\partial^2 \psi'}{\partial t^2} - \nabla^2 \psi'. \tag{23}$$

Equations (22) and (23) define a linear homogeneous system for T' and ψ. It is fourth order in space and third order in time. Supplemented by four homogeneous boundary conditions, it defines an eigenvalue problem for β, the growth rate of solutions proportional to $e^{\beta t}$. Regularity of the solutions at the center, $r=0$, of the fluid implicitly provide two of these boundary conditions, so that only two further boundary conditions need be applied at the boundary, $r=1$, of the fluid. As stated above, we shall focus on the case in which the boundaries are fixed stationary impermeable walls maintained at constant fixed temperature, so that

$$v_r = T' = 0 \quad \text{at} \quad r = 1. \tag{24}$$

Nevertheless, we shall first consider the simpler, though inappropriate, case

$$T' = 0 \quad \text{and} \quad p' = 0 \quad \text{at} \quad r = 1, \tag{25}$$

because these are conditions that lead to the solutions usually presented in textbooks.

III. DISPERSION RELATIONSHIPS

Equations (21) and (23) admit solutions proportional to

$$T' = j_n(kr) S_n(\theta, \phi) e^{\beta t},$$

$$\psi' = - \frac{(\beta + \epsilon^2 \gamma k^2)}{(\gamma - 1) k^2} j_n(kr) S_n(\theta, \phi) e^{\beta t}, \tag{26}$$

or equivalently

$$T' = j_n(kr)S_n(\theta,\phi)e^{\beta t},$$

$$\psi' = \frac{\beta}{(\beta^2+k^2)} j_n(kr)S_n(\theta,\phi)e^{\beta t}, \tag{27}$$

where $j_n(z)$ is the spherical Bessel function of order n, $S_n(\theta,\phi)$ is any surface harmonic of order n, and k and β are constants; (r,θ,ϕ) are spherical coordinates. These equations also show that

$$\epsilon^2\gamma k^4 + \beta\gamma(1+\epsilon^2\beta)k^2 + \beta^3 = 0. \tag{28}$$

Consider first the implications of conditions (25). They imply ρ' $(=p'-T')$ and by Eq. (21) $\nabla^2\psi'$ vanish on $r=1$. Therefore, by Eq. (23), $\psi'(1)=0$. This and the first part of Eq. (25) show that the eigenfunction is a single mode of the form (26) and (27), with k any nonzero root of

$$j_n(k) = 0. \tag{29}$$

Without loss of generality, we may consider only non-negative roots. The first three are listed for $n=0-11$ on p. 467 of [2]; for $n=0-2$ they are (approximately in the case of $n=1,2$)

$$k = \pi,\ 2\pi,\ 3\pi,\ 4\pi, \ldots \quad \text{for}\ n=0, \tag{30}$$

$$k \approx 4.493,\ 7.725,\ 10.904,\ 14.066, \ldots \quad \text{for}\ n=1, \tag{31}$$

$$k \approx 5.763,\ 9.095,\ 12.323,\ 15.515, \ldots \quad \text{for}\ n=2, \tag{32}$$

$$\ldots \tag{33}$$

Equation (28) is now a cubic equation for β that directly determines three possible eigenfrequencies. Consistent with physical expectations, none of these has a positive real part. This is perhaps most easily confirmed by applying Routh's rule and by recalling that necessarily $\gamma>1$. For $1<\gamma<9$, two of the roots of Eq. (28) are complex for all k and ϵ. In what follows, we shall be mainly interested in the case $\epsilon \ll 1$. By Eq. (28), the eigenfrequencies are then approximately

$$\beta = \pm \imath\gamma^{1/2}k - \frac{1}{2}(\gamma-1)k^2\epsilon^2, \tag{34a}$$

$$\beta = -k^2\epsilon^2. \tag{34b}$$

In dimensional units, Eqs. (34a) describe lightly damped oscillations of frequency kc, where $c=\gamma^{1/2}c_T$ is the adiabatic sound speed. These high frequency "acoustic modes" are not our main concern. Equation (34b) describes slow aperiodic decay on the thermal time scale $1/\chi k^2$ and is the thermal mode we seek. The eigenfunctions (26) or (27) give

$$T' = j_n(kr)S_n(\theta,\phi)e^{\beta t},$$

$$v_r = -\frac{\beta k}{(\beta^2+k^2)} j_n'(kr)S_n(\theta,\phi)e^{\beta t}. \tag{35}$$

Since v_r is generally nonzero on $r=1$, the boundary moves radially.

We now abandon Eq. (25) and consider the implications of the more realistic boundary conditions (24). The analysis is a little more complicated. For the putative β, Eq. (28) is a quadratic equation for k^2, determining $\pm k_1$ and $\pm k_2$, but only two of these, k_1 and k_2 (say), give independent solutions; the results that follow are unchanged if k_1 and/or k_2 are reversed in sign. We shall refer to the solutions derived from k_1 and k_2 as "mode 1" and "mode 2." In general both of these are required in order to satisfy Eq. (24). We therefore now take

$$T' = [T_1 j_n(k_1 r) + T_2 j_n(k_2 r)]S_n(\theta,\phi)e^{\beta t}, \tag{36}$$

where T_1 and T_2 are constants. Corresponding to the two alternatives (26) and (27), we then have

$$v_r = \left[T_1\frac{(\beta+\epsilon^2\gamma k_1^2)}{(\gamma-1)k_1} j_n'(k_1 r) + T_2\frac{(\beta+\epsilon^2\gamma k_2^2)}{(\gamma-1)k_2} j_n'(k_2 r)\right] \times S_n(\theta,\phi)e^{\beta t}, \tag{37}$$

$$v_r = -\left[T_1\frac{\beta k_1}{(\beta^2+k_1^2)} j_n'(k_1 r) + T_2\frac{\beta k_2}{(\beta^2+k_2^2)} j_n'(k_2 r)\right]S_n(\theta,\phi)e^{\beta t}. \tag{38}$$

These alternatives give two equivalent forms of the eigenvalue condition that Eq. (24) imply:

$$(\beta+\epsilon^2\gamma k_1^2)\frac{j_n'(k_1)}{k_1 j_n(k_1)} = (\beta+\epsilon^2\gamma k_2^2)\frac{j_n'(k_2)}{k_2 j_n(k_2)}, \tag{39}$$

$$\frac{1}{(\beta^2+k_1^2)}\frac{k_1 j_n'(k_1)}{j_n(k_1)} = \frac{1}{(\beta^2+k_2^2)}\frac{k_2 j_n'(k_2)}{j_n(k_2)}. \tag{40}$$

Sometimes the analysis that follows is more transparent using Eq. (39) rather than Eq. (40), and sometimes the reverse.

In what follows, we shall examine the consequences, Eqs. (39) and (40), of Eq. (24) for $\epsilon \ll 1$. In that case it is found that $|\beta|$ is small compared with ϵ^{-2}. The approximate solutions of Eq. (28) are then

$$k_1^2 = -\frac{\beta^2}{\gamma} + \frac{(\gamma-1)}{\gamma^2}\beta^3\epsilon^2 + \cdots,$$

$$k_2^2 = -\frac{\beta}{\epsilon^2} - \frac{(\gamma-1)}{\gamma}\beta^2 + \cdots, \tag{41}$$

from which it follows that

$$k_1 = \pm\imath\left[\frac{\beta}{\gamma^{1/2}} - \frac{(\gamma-1)}{2\gamma^{3/2}}\beta^2\epsilon^2 + \cdots\right],$$

$$k_2 = \pm \imath\left[\frac{\beta^{1/2}}{\epsilon} + \frac{(\gamma-1)}{2\gamma}\beta^{3/2}\epsilon + \cdots\right]. \quad (42)$$

These appear to be expansions in powers of ϵ^2 but, since β will be expanded in powers of ϵ below, they too are expansions in powers of ϵ.

IV. HIGH FREQUENCY (ACOUSTIC) MODES

In this section we investigate the high frequency solutions of the system, i.e., the two modes for which $\beta = O(1)$. For these, $k_1 = O(1)$ and $k_2 = (\epsilon^{-1})$. To leading order Eqs. (39) or (40) give

$$j_n'(k_1) = O(k_2^{-1}) = O(\epsilon) \quad (43)$$

so that

$$k_1 = k_{10} + \epsilon k_{11} + O(\epsilon^2) \quad (44)$$

where

$$j_n'(k_{10}) = 0. \quad (45)$$

The first three roots of Eq. (45) are listed for $n=0-13$ on p. 468 of [2]; for $n=0-2$ they are approximately

$$k_{10} \approx 1.166,\ 4.604,\ 7.790,\ 10.950, \ldots \quad \text{for } n=0, \quad (46)$$

$$k_{10} \approx 2.461,\ 6.029,\ 9.261,\ 12.445, \ldots \quad \text{for } n=1, \quad (47)$$

$$k_{10} \approx 3.633,\ 7.367,\ 10.664,\ 13.883, \ldots \quad \text{for } n=2, \quad \ldots \quad (48)$$

By Eqs. (44) and (42)

$$\beta = \mp \imath\gamma^{1/2}k_1 + O(\epsilon^2) = \mp \imath\gamma^{1/2}k_{10} \mp \imath\gamma^{1/2}k_{11}\epsilon + O(\epsilon^2). \quad (49)$$

By Eq. (45), it also follows by Taylor expansion that

$$\frac{j_n'(k_1)}{k_1 j_n(k_1)} \approx \frac{j_n'(k_{10})}{k_{10}j_n(k_{10})} + \epsilon k_{11}\left[\frac{j_n'(k)}{kj_n(k)}\right]_{k=k_{10}} = \epsilon k_{11}\left[\frac{j_n''(k)}{kj_n(k)}\right]_{k=k_{10}} = -\epsilon\frac{k_{11}}{k_{10}}\left(1 - \frac{n(n+1)}{k_{10}^2}\right). \quad (50)$$

For all positive roots of Eq. (45), $k_{10} > n + \frac{1}{2}$; see [[3], Sec. 15.3]. It follows that $k_{10}^2 > n(n+1)$, so that the quantity in parenthesis in Eq. (50) is positive. We choose the root k_2 of Eq. (42) with a positive imaginary part:

$$k_2 = \frac{(\pm 1 + \imath)}{\epsilon}\left(\frac{\gamma k_{10}^2}{4}\right)^{1/4} + O(1). \quad (51)$$

Since $\mathrm{Im}(k_2) > 0$ and $|k_2| \gg 1$,

$$\frac{j_n'(k_2)}{k_2 j_n(k_2)} = -\frac{\imath}{k_2} + O(\epsilon^2) = \frac{(-1 \mp \imath)}{(4\gamma k_{10}^2)^{1/4}}\epsilon + O(\epsilon^2). \quad (52)$$

It now follows from Eq. (39) that

$$\frac{k_{11}}{k_{10}}\left(1 - \frac{n(n+1)}{k_{10}^2}\right) = (\gamma-1)\frac{(-1 \mp \imath)}{(4\gamma k_{10}^2)^{1/4}}, \quad (53)$$

i.e.,

$$k_{11} = (-1 \mp \imath)(\gamma - 1)\left(\frac{k_{10}^2}{4\gamma}\right)^{1/4}\left(1 - \frac{n(n+1)}{k_{10}^2}\right)^{-1}, \quad (54)$$

so that, by Eq. (49),

$$\beta = \mp \imath\gamma^{1/2}k_{10} + (-1 \pm \imath)(\gamma - 1)\left(\frac{\gamma k_{10}^2}{4}\right)^{1/4} \times \left(1 - \frac{n(n+1)}{k_{10}^2}\right)^{-1}\epsilon. \quad (55)$$

As expected, $\mathrm{Re}(\beta) < 0$.

The approximate form of the eigenfunction can be found from Eqs. (36) and (37) or (38). To leading order

$$v_r = \mp \frac{\imath\gamma^{1/2}}{(\gamma-1)}\frac{j_n'(k_{10}r)}{j_n(k_{10})}S_n(\theta,\phi)e^{\pm\imath\gamma^{1/2}k_{10}t} \quad (56)$$

is uniformly valid but T' exhibits a boundary layer at $r=1$, the first term in its (composite) expansion being

$$T' = \left[\frac{j_n(k_{10}r)}{j_n(k_{10})} - e^{\imath k_2(1-r)}\right]S_n(\theta,\phi)e^{\pm\imath\gamma^{1/2}k_{10}t}, \quad (57)$$

where k_2 is given by Eq. (51).

V. LOW FREQUENCY (PROSPERETTI) MODE

In this section we investigate the low frequency solution of the system. For this, $\beta = O(\epsilon^2)$, so that $k_1 = O(\epsilon^2)$ and $k_2 = O(1) \gg k_1$. The modes $n=0$ and $n \neq 0$ require separate treatment. This is because

$$\frac{k_1 j_n'(k_1)}{j_n(k_1)} \sim \begin{cases} -\frac{1}{3}k_1^2 & \text{for } k_1 \to 0 \text{ and } n=0, \\ n & \text{for } k_1 \to 0 \text{ and } n \neq 0. \end{cases} \quad (58)$$

A. Radial modes, $n=0$

By Eq. (58), both Eqs. (39) and (40) give, to leading order,

$$(\gamma-1)\frac{k_2 j_0'(k_2)}{j_0(k_2)} = \frac{\beta}{\epsilon^2}\frac{j_0'(k_1)}{k_1 j_0(k_1)} = -\frac{\beta}{3\epsilon^2} = \tfrac{1}{3}k_2^2 \quad (59)$$

or

$$(\gamma-1)\left(\cot k_2 - \frac{1}{k_2}\right) = \tfrac{1}{3}k_2, \quad (60)$$

the roots of which are

$$k_2 \approx 3.591,\ 6.566,\ 9.625, \ldots \quad \text{for } \gamma = \tfrac{5}{3}, \quad (61)$$

$$k_2 \approx 3.448,\ 6.462,\ 9.548, \ldots \quad \text{for } \gamma = \tfrac{7}{5}. \quad (62)$$

Therefore

$$\beta/\epsilon^2 \approx -12.894, \quad -43.118, \quad -92.649, \ldots \quad \text{for} \quad \gamma = \tfrac{5}{3}, \tag{63}$$

$$\beta/\epsilon^2 \approx -11.887, \quad -41.755, \quad -99.168, \ldots \quad \text{for} \quad \gamma = \tfrac{7}{5}. \tag{64}$$

These values may be contrasted with the results obtained if the procedure outlined on p. 207 of [1] is followed: by Eqs. (33), (34a), and (34b), these are, for all γ,

$$k_2 = \pi, 2\pi, 3\pi, \ldots,$$

$$\beta/\epsilon^2 \approx -\pi^2, \quad -4\pi^2, \quad -9\pi^2, \ldots. \tag{65}$$

To leading order, the eigenfunctions corresponding to solutions of Eq. (60) are proportional to

$$T' = \left[1 - \frac{j_0(k_2 r)}{j_0(k_2)}\right] e^{\beta t}, \tag{66}$$

$$v_r = \frac{\epsilon^2 k_2}{(\gamma-1)}\left[(\gamma-1)\frac{j_0'(k_2 r)}{j_0(k_2)} - \tfrac{1}{3}k_2 r\right] e^{\beta t}. \tag{67}$$

B. Nonradial modes, $n \neq 0$

The frequencies of the radial modes depended on γ, albeit weakly. Those of the nonradial modes are independent of γ to leading order. By writing Eq. (40) as

$$(\beta^2 + k_2^2)\frac{j_n(k_2)}{k_2 j_n'(k_2)} = (\beta^2 + k_1^2)\frac{j_n(k_1)}{k_1 j_n'(k_1)} \to 0 \quad \text{for} \quad \beta \to 0, \tag{68}$$

it is immediately apparent that, to leading order,

$$j_n(k_2) = 0. \tag{69}$$

This is the same dispersion relationship as was obtained for the illustrative example discussed in Sec. IV, and the values of k_2 and β/ϵ^2 are again given by Eqs. (33) and (34b), the $n=0$ modes being ignored since they were dealt with in the last section. To leading order, the eigenfunctions are proportional to

$$T' = j_n(k_2 r) S_n(\theta,\phi) e^{\beta t},$$

$$v_r = \epsilon^2 k_2 [j_n'(k_2 r) - j_n'(k_2) r^{n-1}] S_n(\theta,\phi) e^{\beta t}. \tag{70}$$

The T' eigenfunction is the same as in Eq. (35) but obviously v_r is different since it must vanish on $r=1$.

VI. INTERPRETATION

One objective of this section is to demonstrate that the results of Secs. V A and V B for the low frequency modes may be derived from an approximation devised by Prosperetti [4]. We also aim to show how that approximation leads naturally to Eq. (3). We revert to dimensional units.

The relation

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial \rho}\right)_T \frac{d\rho}{dt} + \left(\frac{\partial p}{\partial T}\right)_\rho \frac{dT}{dt} = c_T^2 \frac{d\rho}{dt} + c_T^2 \alpha\rho \frac{dT}{dt}, \tag{71}$$

may, by Eqs. (4) and (12), be written as

$$\frac{dp}{dt} = -c^2 \rho [\nabla \cdot \mathbf{v} - \alpha\chi \nabla^2 T]. \tag{72}$$

(We here made use of the relations $c_T^2 = c^2/\gamma$ and $c_p = c_v + \alpha^2 T c_T^2$.) Prosperetti [4] argues that, at low frequencies, the pressure has time approximately to equalize across the system, i.e., p is almost uniform in space though it remains time-dependent. This implies that Eq. (72) can be replaced by

$$\nabla \cdot [c^2 \rho (\mathbf{v} - \alpha\chi \nabla T)] = -\dot{p} = -\nabla \cdot (\dot{p}\mathbf{r}/3), \tag{73}$$

where $\dot{p} = \partial p/\partial t$. Then, for irrotational flows,

$$\mathbf{v} = \alpha\chi \nabla T - \frac{\dot{p}}{3c^2\rho}\mathbf{r} + \nabla\phi, \quad \text{or} \quad \psi' = -\alpha\chi T + \frac{\dot{p} r^2}{6c^2\rho} - \phi, \tag{74}$$

where $\nabla^2\phi = 0$. If $r = R(\theta,\phi,t)$ is the boundary, Eq. (74) gives

$$\dot{R} = \alpha\chi\left(\frac{\partial T}{\partial r}\right)_{r=R} + \frac{\partial\phi}{\partial r} - \frac{\dot{p}}{3c^2\rho}R. \tag{75}$$

Then, by Eq. (74),

$$\mathbf{v} = \alpha\chi \nabla T + \left[\dot{R} - \alpha\chi\left(\frac{\partial T}{\partial r}\right)_{r=R} - \left(\frac{\partial\phi}{\partial r}\right)_{r=R}\right]\frac{\mathbf{r}}{R} + \nabla\phi, \tag{76}$$

$$\nabla \cdot \mathbf{v} = \alpha\chi\nabla^2 T + \frac{3}{R}\left[\dot{R} - \alpha\chi\left(\frac{\partial T}{\partial r}\right)_{r=R} - \left(\frac{\partial\phi}{\partial r}\right)_{r=R}\right]. \tag{77}$$

Substituting these results into Eq. (12), and ignoring terms of order $(\nabla T)^2$ associated with entropy production [similar to terms already omitted from Eq. (10)], we obtain

$$\frac{\partial T}{\partial t} + \frac{\dot{R}}{R} r \frac{\partial T}{\partial r} = \chi\nabla^2 T - \frac{3(\gamma-1)}{\alpha R}\left[\dot{R} - \alpha\chi\left(\frac{\partial T}{\partial r}\right)_{r=R} - \left(\frac{\partial\phi}{\partial r}\right)_{r=R}\right]. \tag{78}$$

We apply Eq. (78) to the eigenvalue problem posed in Sec. III, for which conditions (24) hold at $r=R$ =const and $\dot{R}=0$. For the radial modes ($n=0$), the only nonsingular solution of $\nabla^2\phi=0$ is an inconsequential constant, and Eq. (78) reduces to

$$\frac{\partial T'}{\partial t} = \chi\nabla^2 T' + \frac{3\chi(\gamma-1)}{R}\left(\frac{\partial T'}{\partial r}\right)_{r=R}. \tag{79}$$

For $T' \propto e^{\beta t}$, where β is now dimensional, this gives

$$\frac{d^2(rT')}{dr^2} + k^2(rT') = -\frac{3(\gamma-1)}{R}\left(\frac{\partial T'}{\partial r}\right)_{r=R} r, \tag{80}$$

where $k^2 = -\beta/\chi$. The required nonsingular solution satisfying $T'(R)=0$ is

$$T' = -\frac{3(\gamma-1)}{k^2R}\left(\frac{\partial T'}{\partial r}\right)_{r=R}\left[1-\frac{\sin kr}{kr}\bigg/\frac{\sin kR}{kR}\right]. \quad (81)$$

On differentiating this result, we obtain

$$\frac{\partial T'}{\partial r} = \frac{3(\gamma-1)}{\sin kR}\left(\frac{\partial T'}{\partial r}\right)_{r=R}\left[\frac{\cos kr}{kr}-\frac{\sin kr}{(kr)^2}\right]. \quad (82)$$

Applying this result at $r=R$, we recover the dimensional form of Eq. (60).

For the nonradial modes ($n\neq 0$), Eq. (75) gives

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=R} - \alpha\chi\left(\frac{\partial T'}{\partial r}\right)_{r=R} = 0, \quad (83)$$

and Eqs. (78) and (76) become

$$\frac{\partial T'}{\partial t} = \chi\nabla^2 T',$$

$$\mathbf{v} = \alpha\chi\nabla T' + \nabla\phi, \quad (84)$$

which gives

$$T' = j_n(kr)S_n(\theta,\phi)e^{\beta t}, \quad \text{where} \quad j_n(kR)=0. \quad (85)$$

The nonsingular solution of $\nabla^2\phi=0$ is proportional to $r^nS_n(\theta,\phi)e^{\beta t}$ and Eqs. (83) and (85) give

$$\phi = -n^{-1}\alpha\chi kR j_n'(kR)(r/R)^nS_n(\theta,\phi)e^{\beta t} \quad (86)$$

so that, by Eq. (84),

$$v_r = \alpha\chi k[j_n'(kr) - j_n'(kR)(r/R)^{n-1}]S_n(\theta,\phi)e^{\beta t}. \quad (87)$$

Equations (85) and (87) are the dimensional forms of Eqs. (69) and (70).

VII. CONCLUSIONS

We were led to Eq. (3) during efforts to understand the temperature inside a collapsing bubble of gas that is surrounded by a fluid; see Hopkins *et al.* [5]. These bubbles expand during the rarefaction phase of a sound field and collapse during the ensuing compression. The time scale for the expansion may be much longer than the time scale of the collapse. As a consequence, the expansion may be essentially isothermal, whereas part of the collapse may be nearly adiabatic. The transition from the isothermal to the adiabatic phase has a significant effect on the temperature reached inside the bubble at its minimum radius, which is when the flash of light is emitted that gives sonoluminescence its name. To gain insight into the isothermal to adiabatic transition, we have employed the approach of Prosperetti [4] to thermal conduction in a pulsating bubble. In the limit in which pressure variations are neglected, he arrived at a solution that, in dimensional variables, is

$$\mathbf{v} = \frac{\dot{R}}{R}\mathbf{r} + \frac{\alpha\kappa}{\rho c_p}\left[\boldsymbol{\nabla}T - \left(\frac{\partial T}{\partial r}\right)_{r=R}\frac{\mathbf{r}}{R}\right]. \quad (88)$$

It may be seen that, even when $\dot{R}=0$, there is a radial flow in the bubble that, when substituted into Eq. (12), yields the deviations from Eq. (1) that are contained in Eq. (3), and that this accounts for the deviations between Eqs. (61) and (64) and the textbook result (65).

The technique described in this paper may also find applications in other areas of physics, such as mass diffusion and vorticity dispersion. Our method can also be applied to other than spherical geometries, for example, to systems possessing cylindrical symmetry. It remains to be seen whether such generalizations are significant.

ACKNOWLEDGMENTS

S.P. thanks the ONR for support and Brian Kappus and Carlos Camara for valuable discussions. P.R. thanks the NSF for support under Grant No. EAR-0609778.

[1] L. D. Landau and L. D. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, UK, 1987).
[2] *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, DC, 1964).
[3] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University Press, Cambridge, England, 1944).
[4] A. Prosperetti, J. Fluid Mech. **222**, 587 (1991).
[5] S. D. Hopkins, S. J. Putterman, B. A. Kappus, K. S. Suslick, and C. G. Camara, Phys. Rev. Lett. **95**, 254301 (2005).